

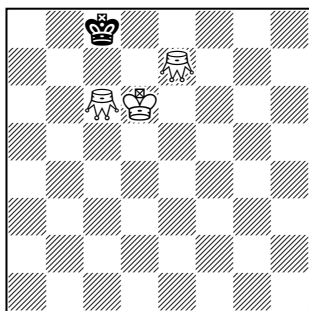
# KING AND TWO FERSES AGAINST KING

analysis initiated by Noam Elkies

Last August, Noam Elkies sent me a remarkable conjecture: if we count stalemate as a win, cannot White force a win with just *two* ferses against a bare king? Furthermore, can he not do this even if the ferses run on squares of the same colour?

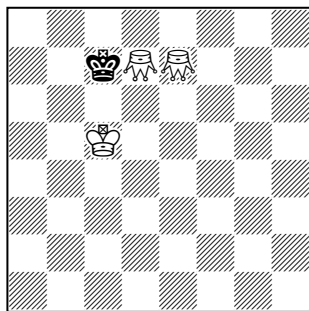
To put matters bluntly, I did not believe it. So I sent him a position where I could not make progress, and back came a winning method; then I sent him another, and back came proof of another win; and a few weeks later, he analysed the ending by computer, and even without taking into account the possibility that White might be able to sacrifice one fers and stalemate with the other (ignoring this possibility allowed an existing program to be adapted with minimum alteration) he confirmed that it was indeed a general win whichever sets of squares the ferses ran on.

This established the general result, and placed upper bounds for the lengths of the longest wins in the two cases (ferses on squares of the same and of opposite colours). There were however positions where White could speed things up by sacrificing one fers in order to stalemate with the other. Consider this position, for which Noam insists I take the credit as between ourselves (others may well have discovered it earlier) :



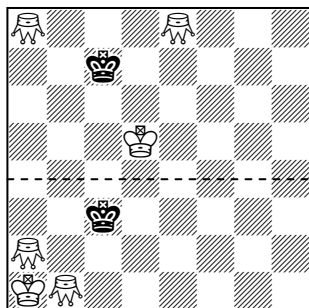
Here, White can stalemate in five by playing **1 Fd8**, since 1...Kxd8 allows 2 Fd7 stalemating at once (if instead 1...Kb8 then 2 Fc7+ Ka8/Kc8 3 Fb6 etc, or 2...Ka7 3 Kc5). Noam then

drew attention to this position :



Here, White can stalemate in seven only by sacrificing a fers, though he may choose which one to give away: **1 Fc8 Kb8** (trying to delay matters) **2 Kb6 Kxc8** (no choice now) **3 Kc6** etc, or **1 Fc6 Kb8** (1...Kc8 2 Kb6 and stalemate next move) **2 Fd8 Kc8** **3 Kd6** transposing into the previous example. To stalemate without a sacrifice takes thirteen moves.

We then looked for same-colour positions in which sacrifices by White might be beneficial, and Noam sent me a series of examples culminating in the upper position below :

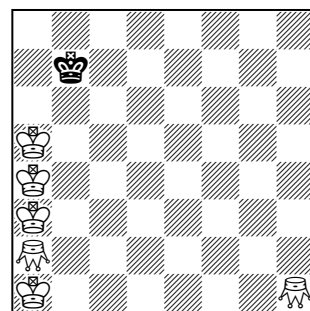


Black to play is soon seen to lose: 1...Kb8 2 Kc6 Kxa8 3 Kb6 Kb8 4 Fd7 etc, or 1...Kd8 2 Ff7 Ke7 3 Fe6 and White is secure, or here 2...Kc7 3 Fe6 and 3...Kb8 again fails to 4 Kc6 etc. But White to play cannot win: 1 Kc5 Kd8 and Fe8 will be hunted down, or 1 Ke6 Kb8 and the White king is too far away to trap the Black, or 1 Ff7 Kb8 and it is the White fers which is too far away. Noam's computer analysis had already identified one reciprocal zugzwang in the shape of the lower position, but this new one was much more interesting.

We therefore felt that the time had come for a definitive computer analysis including sacrifices. This time, Noam left the programming to

me, partly so that his results would receive independent confirmation. My analysis found the same longest-win positions as Noam's, verified the two reciprocal zugzwangs above, and identified thirteen more, some even more remarkable than Kd5, Fa8/e8 v Kc7. The rest of the article will therefore be couched in terms of this definitive analysis. However, the situation is very much the same as in the case of the analysis of the as-Suli position which I mentioned in last time's review of *Scacchia Ludus*: my computer analysis may have dotted the final "i", but it was wholly routine and derivative, and it is the person who had opened up the field who deserves the credit.

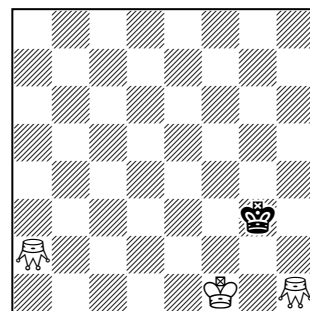
The longest wins occur in the same-colour case, and there are four maximal positions differing only in the location of the White king :



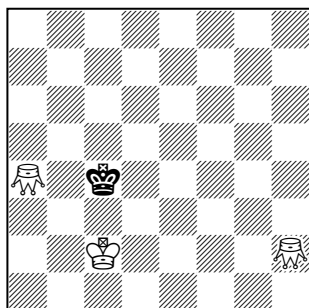
Black to play, White stalemates in 44 (wKa5 assumed in what follows)

White cannot usefully sacrifice a fers in the play from these positions, and the optimal without-sacrifice line sent by Noam carries across unchanged (there are equi-optimal alternatives at various points). The lines from all four positions converge at 4 Ke1.

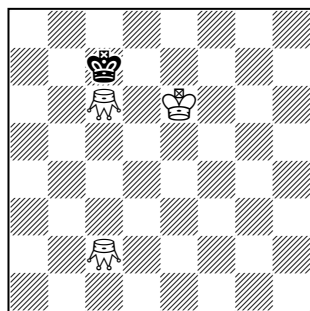
**0...Kc6 1 Kb4 Kd5 2 Kc3 Ke4 3 Kd2 Kf3 4 Ke1 Kg3 5 Kf1 :**



White has saved his fers on h1, and can gather his men and advance.  
**5...Kf3 6 Fb1 Ke3 7 Ke1 Kd3 8 Fg2 Kc3 9 Kd1 Kd3 10 Fc2+ Ke3 11 Ke1 Kd4 12 Kd2 Kc4 13 Ke3 Kd5 14 Kd3 Kc5 15 Ke4 Kb4 16 Kd4 Kb5 17 Kd5 Ka5 18 Kc5 Ka6 19 Ff3 Kb7 20 Kd6 Kc8 21 Fe4 Kd8 22 Fd5 Ke8 23 Ke6 Kf8 24 Kf6 Ke8 25 Fc6 Kd8 26 Ke6 Kc7 :**

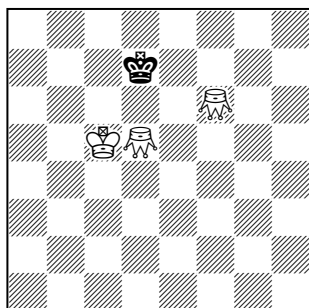


Now 6...Kb4 will be met by 7 Fb3, but Black can still make a nuisance of himself by attacking each fers in turn.  
**6...Kd4 7 Fg1 Ke3 8 Kd1 Kf3 9 Ke1 Ke3 10 Fb3 Kd3 11 Kd1 Ke3 12 Fc2 Kf3 13 Ke1** (White has finally secured his forces, and can start the advance) **Ke3 14 Ff2+ Kd4 15 Kd2 Kc4 16 Ke3 Kd5 17 Fg3 Kc4 18 Ff4 Kc5 19 Fd3 Kd5 20 Fe4+ Kc4 21 Fe5 Kc5 22 Kd3 Kb5 23 Kd4 Kc6 24 Kc4 Kd7 25 Ff5 Kc6 26 Fe6 Kb6 27 Fd5 Kc7 28 Kc5 Kd7 29 Ff6 :**

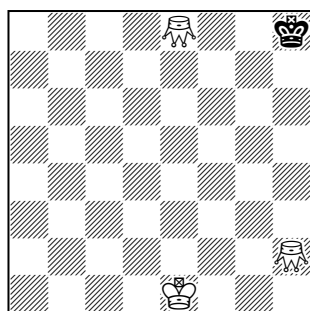


**27 Kd5 Kd8 28 Kd6 Ke8 29 Ke6 Kf8 30 Kf6 Ke8 31 Fb3 Kd8 32 Ke6 Kc7 33 Fb5 Kb7 34 F3c4 Kc8 35 Fd5 Kc7 36 Fdc6 Kd8 37 Kf7 Kc8 38 Ke8 Kc7 39 Ke7 Kc8 40 Kd6 Kb8 41 Kd7 Ka7 42 Kc7 Ka8 43 Kb6 Kb8 44 Fb7 stalemate.**

The starting position of the longest win in the opposite-colour case is unique :



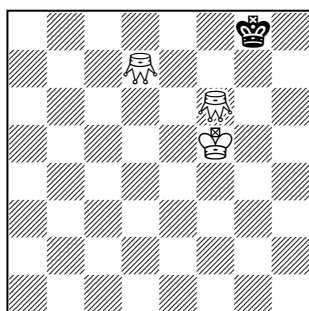
**29...Kd8** (29...Ke8 lets White get there one move sooner, 30 Fe6 Kf8 31 Kd5) **30 Fe6 Ke8 31 Kd6 Kf8 32 Kd5! Ke8 33 Ke5 Kf8 34 Fd7 Kf7 35 Kf5 Kg8 :**



White to play stalemates in 41

This is an unusual configuration for the starting position of the longest win in a “generally won” ending, but there is a prelude where, in Noam’s words, “Black nearly manages to corral a fers à la Réti”, and after Black’s fourth move we have a more typical position where Black has the centre and White’s men are scattered round the edge.

**1 Fd7 Kg7 2 Fe6 Kf6 3 Fb5 Ke5 4 Fa4 Kd4 5 Kd2 Kc4 6 Kc2 :**

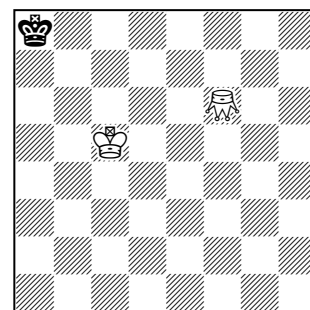


Now we are on familiar ground: **36 Fe8 Kf8 37 Ke6! Kg8** (37...Kxe8 38 Fe7 stalemate) **38 Ff7+ Kf8 39 Fg6 Kg8 40 Ke7 Kh8 41 Kf8 stalemate.**

Without the sacrifice 37 Ke6, the final stage would take six moves longer (an optimal non-sacrifice line is **36 Kg6 Kf8 37 Kh5 Kg8 38 Kh6 Kf7 39 Kg5 Kg8 40 Fe7 Kf7 41 Fd6 Kg7 42 Fe6 Kh7 43 Kh5 Kg7 44 Fe7 Kh7 45 Ff6 Kg8 46 Kg6 Kf8 47 Ff7**). However, Noam’s original analysis had found the longest win without a sacrifice to take 46 moves, not 47, and the reason is to be found in the play at moves 29-32. With the aid of the sacrifice 37 Ke6, White’s 32 Kd5 forces stalemate at move 41. Without a sacrifice, 32 Kd5 leads to stalemate only at move 47, but White has an alternative move 32 Kd7 which forces it at move 45. Black therefore does better to play **29...Ke8**, since after **30 Fe6 Kf8** the shortening move Kd7 is not available. White has nothing better than **31 Kd5**, and stalemate follows at move 46.

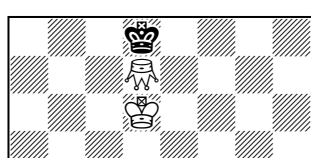
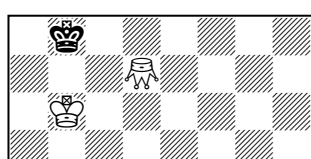
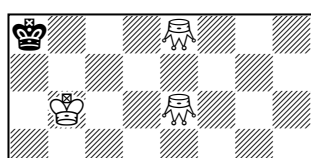
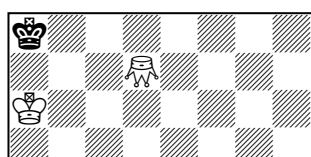
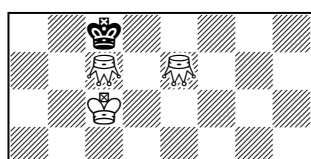
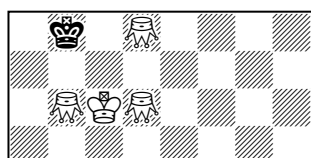
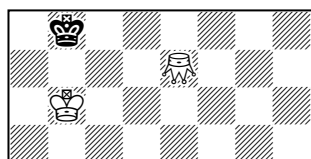
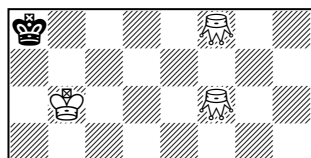
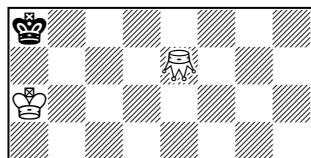
Now let us look at the reciprocal zugzwangs. Normally, any reciprocal zugzwangs that exist in a generally won ending are typified by the lower position in the middle of the previous page: the stronger side is constrained, and cannot disentangle itself. The upper position shows a different mechanism. Black is able to capture a helpless fers on a8, and everything depends on whether White can force stalemate with his remaining fers.

We therefore need to look at endings with a single fers. White can force stalemate only if the Black king is already on the edge, and a typical longest stalemate takes seven moves :



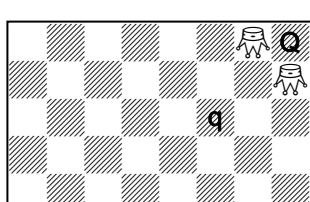
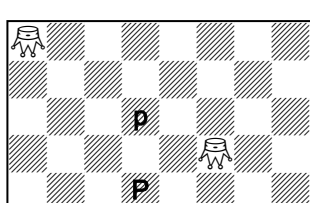
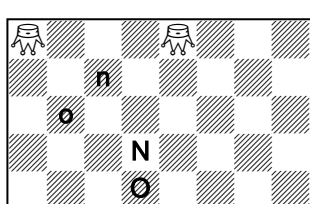
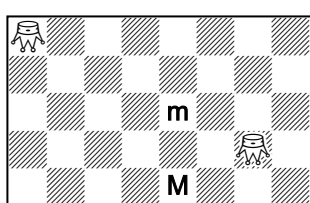
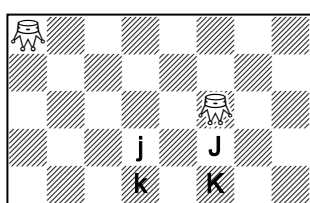
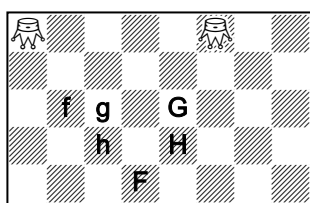
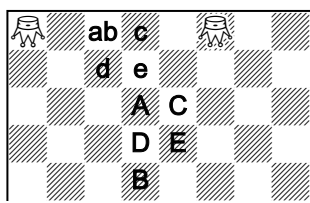
The play from here goes **1 Kb6 Kb8 2 Fe7 Kc8 3 Kc6 Kb8 4 Fd8/Fd6 Ka8** (best) **5 Kb6 Kb8 6 Fe7+** etc, and each of the positions after 1 Kb6, 2 Fe7, 3 Kc6, and 4 Fd8/Fd6 is reciprocal zugzwang. There are in fact

fourteen positions of reciprocal zugzwang with a single fers :



It is easily verified that in each case White can force an eventual stalemate (or has given one) if Black is to play, but cannot if he himself is to play.

With two ferses, there are fifteen positions of reciprocal zugzwang :



The lower case letter gives the Black king, the upper case letter the White. If the Black king is on c8, there are two possible positions for the White, d6 and d4. **Nn** and **Qq** are the two positions which we saw on page 90.

I sent these to Noam while they were still hot from the oven, with an initial note that he might find **Aa** of interest and a subsequent note that **Bb** seemed even more remarkable: how could a position like this, with the White king isolated on the distant square d4, possibly be reciprocal zugzwang? But as Noam said in reply, the nice thing about having a complete database is that it can not only raise such questions but also answer them. Once we have put the two sets of positions side by side, all becomes clear.

Consider **Aa**. Black to play must go for a8, but after 1...Kb8 2 Kc6 Kxa8 3 Kb6 we have one of the reciprocal zugzwangs in the first column. White to play might seem to have 1 Fe7, but after 1...Kb8 2 Kc6 Kxa8 3 Kb6 his fers is on e7 instead of f8, and now 3...Kb8 leaves him as the player in zugzwang. Alternatively, try 1 Kc6: no, 1...Kd8 threatens to hunt down the fers on f8, forcing 2 Kd6, and 2...Kc8 repeats the original position.

Now consider **Bb**. 1...Kb8 2 Kc5 etc, 1...Kc7 2 Kd5, 1...Kd7 2 Ke5. 1...Kd8, to hunt down the fers on f8? No, 2 Fg7, when 2...Ke7/Ke8 is met by 3 Ke5 and 4 Ff6. But White to play has no good move, despite the isolation of his king: 1 Kd5 Kc7, 1 Ke5 Kd7, 1 Kc5 Kd7/Kd8 etc.

There is a lot more, as those who are tempted to explore further will discover. For example, from **Dd**, if Black plays 1...Kd8, White must again play 2 Fg7; if instead 2 Ke6 then 2...Kc7 forces 3 Kd5 repeating the position. It is also instructive to work out why some apparently equivalent positions are *not* reciprocal zugzwang. Or what about Kd4, Fb8/e7 v Kc6, sent to me by Noam while we were still looking for same-colour positions where sacrifices might be beneficial? Black loses after 1...Kb7 2 Kc5 Kxb8 3 Kb6 but draws if he attacks e7 first, 1...Kd7 2 Ff6 Kc8 etc.

Truly, the lowly fers can give rise to positions of remarkable subtlety.